

# Free Arrangements over Finite Field

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## Abstract

The freeness of hyperplane arrangements in a three dimensional vector space over finite field is discussed. We prove that if the number of hyperplanes is greater than some bound, then the freeness is determined by the characteristic polynomial.

## 1 Introduction

A hyperplane arrangement  $\mathcal{A}$  in an  $\ell$ -dimensional vector space  $V$  is a finite set of linear subspaces of codimension one. An arrangement  $\mathcal{A}$  is said to be free when the associated module of logarithmic vector fields is a free module. Study of arrangements of this class was started by K. Saito [Sa] and a remarkable factorization theorem was proved by H. Terao [Te2]. This theorem asserts that the characteristic polynomial  $\chi(\mathcal{A}, t)$  of a free arrangement completely factors into linear polynomials in  $\mathbb{Z}$ . It imposes a necessary condition on the structure of intersection lattice  $L(\mathcal{A})$  for an arrangement  $\mathcal{A}$  to be free. The Terao conjecture is the problem to ask the converse: does the structure of  $L(\mathcal{A})$  characterize freeness of  $\mathcal{A}$ ? This conjecture is still open even in the case  $\ell = 3$ . The purpose of this paper is to propose an affirmative result over finite field  $\mathbb{F}_q$  in the case  $\ell = 3$ . Our main result asserts that if the number of hyperplanes satisfies  $|\mathcal{A}| \geq 2q - 2$ , then  $\mathcal{A}$  is free exactly when  $\chi(\mathcal{A}, q)\chi(\mathcal{A}, q - 1) = 0$ . Proofs are based on Terao's addition-deletion theorem (Theorem 1) and Crapo-Rota's method of counting points by using characteristic polynomials (Theorem 4, Theorem 6).

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## 2 Freeness and characteristic polynomials

Let  $V$  be an  $\ell$ -dimensional linear space over a field  $\mathbb{K}$  and  $S := \mathbb{K}[V]$  be the algebra of polynomial functions on  $V$  that is naturally isomorphic to  $\mathbb{K}[z_1, z_2, \dots, z_\ell]$  for any choice of basis  $(z_1, \dots, z_\ell)$  of  $V^*$ . A (central) hyperplane arrangement  $\mathcal{A}$  is a finite collection of one-codimensional linear subspaces (=hyperplanes) in  $V$ . For each hyperplane  $H$  of  $\mathcal{A}$ , fix a nonzero linear form  $\alpha_H \in V^*$  vanishing on  $H$  and put  $Q := \prod_{H \in \mathcal{A}} \alpha_H$ .

The characteristic polynomial of  $\mathcal{A}$  is defined as

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

where  $L(\mathcal{A})$  is the lattice consisting of intersections of elements of  $\mathcal{A}$ , ordered by reverse inclusion,  $\hat{0} := V$  is the unique minimal element of  $L(\mathcal{A})$  and  $\mu : L(\mathcal{A}) \longrightarrow \mathbb{Z}$  is the Möbius function defined as follows:

$$\begin{aligned} \mu(\hat{0}) &= 1, \\ \mu(X) &= - \sum_{Y < X} \mu(Y), \text{ if } \hat{0} < X. \end{aligned}$$

Fix a hyperplane  $H \in \mathcal{A}$ , we obtain two associated arrangements: deletion  $\mathcal{A}' = \mathcal{A} \setminus \{H\}$  and restriction  $\mathcal{A}'' = H \cap \mathcal{A}'$ . The characteristic polynomials for these arrangements satisfy the following inductive formula

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t).$$

Denote by  $\text{Der}_V$  the  $S$ -module of all polynomial vector fields over  $V$ . For a given arrangement  $\mathcal{A}$ , we define the module of logarithmic vector fields as

$$D(\mathcal{A}) = \{\delta \in \text{Der}_V \mid \delta(\alpha_H) \in \alpha_H S, \forall H \in \mathcal{A}\}.$$

An arrangement  $\mathcal{A}$  is said to be **free**, if  $D(\mathcal{A})$  is a free  $S$ -module, and then the multiset of degrees  $\exp(\mathcal{A}) := (d_1, d_2, \dots, d_\ell)$  of homogeneous basis of  $D(\mathcal{A})$  is called the **exponents**. The following theorems are due to H. Terao.

**Theorem 1** ([Te1], [OT, Thm 4.52])

Let  $\mathcal{A}$  be a non-empty arrangement in  $\mathbb{K}^3$ . Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple as above. Then, any two of the following imply the third:

- $\mathcal{A}$  is free with exponents  $(d_1, d_2, d_3)$ .

- $\mathcal{A}'$  is free with exponents  $(d_1, d_2, d_3 - 1)$ .
- $|\mathcal{A}''| = d_1 + d_2$ .

**Theorem 2** [Te2] If  $\mathcal{A}$  is a free arrangement with exponents  $(d_1, d_2, \dots, d_\ell)$  then the characteristic polynomial is

$$\chi(\mathcal{A}, t) = (t - d_1) \cdots (t - d_\ell).$$

**Example 3** Let  $V$  be an  $\ell$  dimensional vector field over a finite field  $\mathbb{F}_q$  of  $q$  elements and  $\mathcal{A}_{\text{all}}(V)$  be the collection of all hyperplanes in  $V$ . Put

$$\delta_k = \sum_{i=1}^{\ell} x_i^{q^k} \frac{\partial}{\partial x_i}.$$

Then for any linear form  $\alpha$ ,  $\delta_k \alpha = \alpha^k$ . Hence  $\delta_k \in D(\mathcal{A}_{\text{all}})$ . From Saito's criterion [Sa],  $\delta_0, \delta_1, \dots, \delta_{\ell-1}$  form a basis of  $D(\mathcal{A}_{\text{all}})$  and exponents are  $(1, q, \dots, q^{\ell-1})$ .

### 3 Arrangements over finite fields

If  $\mathbb{K}$  is a finite field, the characteristic polynomial  $\chi(\mathcal{A}, t)$  has a special meaning. The following theorem and its proof is found in [OT, 2.69], and it is a special version of more general result obtained in [CR] (see also Theorem 6 below).

**Theorem 4** Let  $\mathcal{A}$  be a hyperplane arrangement in  $V \cong \mathbb{F}_q^\ell$ . Let  $|M(\mathcal{A})|$  denote the cardinality of the complement. Then

$$|M(\mathcal{A})| = \chi(\mathcal{A}, q).$$

This theorem has successfully applied by Athanasiadis [Ath] to compute characteristic polynomials for arrangements defined over  $\mathbb{Q}$ .

An arrangement  $\mathcal{A}$  over a field  $\mathbb{F}_q$  can be naturally considered as an arrangement over an extended field  $\mathbb{F}_{q^k}$ . Since field extensions do not change the intersection lattice, the characteristic polynomial is also unchanged. Hence from Theorem 4, we have

$$\left| V \otimes_{\mathbb{F}_q} \mathbb{F}_{q^k} - \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{F}_{q^k}) \right| = \chi(\mathcal{A}, q^k).$$

Not only  $\chi(\mathcal{A}, q)$ , but also  $\chi(\mathcal{A}, q^k)$  expresses the cardinality of points of complement of arrangement  $\mathcal{A} \otimes \mathbb{F}_{q^k}$ . We have the following lemma.

**Lemma 5** *Let  $\mathcal{A}$ ,  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be arrangements in  $\mathbb{F}_q^\ell$ , then*

- (i)  $\chi(\mathcal{A}, q^k) \geq 0$  for all  $k \in \mathbb{Z}_{>0}$ .
- (ii) If  $\mathcal{A}_1 \subset \mathcal{A}_2$ , then  $\chi(\mathcal{A}_1, q^k) \geq \chi(\mathcal{A}_2, q^k)$  for all  $k \in \mathbb{Z}_{>0}$ .

The following Theorem is due to H. Crapo and G. -C. Rota. It contains Theorem 4 as a special case ( $k = 1$ ). Here we deduce from Theorem 4.

**Theorem 6** [CR, §16, Theorem 1] *Let  $\mathcal{A}$  be an arrangement in  $\mathbb{F}_q^\ell$ . The number of ordered points  $(p_1, \dots, p_k) \in (\mathbb{F}_q^\ell)^k$  satisfying the following condition (\*) is  $\chi(\mathcal{A}, q^k)$ :*

- (\*) *For each hyperplane  $H \in \mathcal{A}$ , there exists at least one point  $p_i$  such that  $p_i \notin H$ .*

**Proof.** Recall that  $\mathbb{F}_{q^k}$  is an  $k$ -dimensional vector space over  $\mathbb{F}_q$ . Let  $x_1, \dots, x_k$  be a  $\mathbb{F}_q$ -basis of  $\mathbb{F}_{q^k}$ . Then the point  $P$  in  $\mathbb{F}_{q^k}^\ell$  is expressed as

$$P = \left( \sum_{j=1}^k a_{1j}x_j, \sum_{j=1}^k a_{2j}x_j, \dots, \sum_{j=1}^k a_{\ell j}x_j \right),$$

where  $a_{ij} \in \mathbb{F}_q$ . Since the defining equation of  $H \otimes \mathbb{F}_{q^k}$  is a linear form with coefficients in  $\mathbb{F}_q$ , the point  $P$  is contained in  $H \otimes \mathbb{F}_{q^k}$  if and only if

$$(a_{1j}, a_{2j}, \dots, a_{\ell j}) \in H, \quad \forall j = 1, \dots, k.$$

Hence  $P$  is in the complement of arrangement  $\mathcal{A} \otimes \mathbb{F}_{q^k}$  if and only if for each  $H \in \mathcal{A}$ , there exists at least one  $j \in \{1, \dots, k\}$  such that  $(a_{1j}, a_{2j}, \dots, a_{\ell j}) \notin H$ . So this gives a bijection between complement of  $\mathcal{A} \otimes \mathbb{F}_{q^k}$  and ordered  $k$  points in  $\mathbb{F}_q^\ell$  satisfying (\*).  $\square$

The next result is shown immediately from Theorem 6, but we give an alternative proof, since the arguments used in the proof is prototypical for later.

**Lemma 7** *Let  $\mathcal{A}$  be an arrangement in  $\mathbb{F}_q^\ell$ . If the characteristic polynomial satisfies  $\chi(\mathcal{A}, q^k) = 0$  for some  $k$ , then  $\chi(\mathcal{A}, q^j) = 0$  for  $0 \leq j \leq k$ .*

**Proof.** The proof is done by induction on the dimension  $\ell$  and “descending” induction on the number  $|\mathcal{A}|$  of hyperplanes. If  $|\mathcal{A}|$  is maximal, in other words,  $\mathcal{A} = \mathcal{A}_{all}(\mathbb{F}_q^\ell)$ , then the characteristic polynomial is

$$\chi(\mathcal{A}_{all}(\mathbb{F}_q^\ell), t) = (t - 1)(t - q) \cdots (t - q^{\ell-1}),$$

so the lemma is trivial. In the case  $\ell = 2$  is also trivial. In general, let  $\mathcal{A}'$  be an arrangement such that  $\mathcal{A}' \subsetneq \mathcal{A}_{all}(\mathbb{F}_q^\ell)$  and assume  $\chi(\mathcal{A}', q^k) = 0$ . We can find a hyperplane  $H$  which is not contained in  $\mathcal{A}'$  and define  $\mathcal{A} = \mathcal{A}' \cup \{H\}$ . From Lemma 5, we have  $\chi(\mathcal{A}', q^k) \geq \chi(\mathcal{A}, q^k) = 0$ . From the inductive hypothesis on the number of hyperplanes, we have

$$\chi(\mathcal{A}, q^j) = 0, \text{ for } 0 \leq j \leq k.$$

Denote  $\mathcal{A}''$  the restriction  $H \cap \mathcal{A}'$ , we have  $\chi(\mathcal{A}'', q^k) = \chi(\mathcal{A}', q^k) - \chi(\mathcal{A}, q^k) = 0$ . Since  $\dim H < \ell$ , we have

$$\chi(\mathcal{A}'', q^j) = 0, \text{ for } 0 \leq j \leq n.$$

Again from inductive formula, we have  $\chi(\mathcal{A}', q^j) = 0$ , for  $0 \leq j \leq n$ .  $\square$

Using the above lemma, we can characterize  $\mathcal{A}_{all}(\mathbb{F}_q^\ell)$ .

**Corollary 8** *Let  $\mathcal{A}$  be an arrangement in  $\mathbb{F}_q^\ell$ . The following conditions are equivalent.*

$$(a) \quad \mathcal{A} = \mathcal{A}_{all}(\mathbb{F}_q^\ell).$$

$$(b) \quad |\mathcal{A}| = \frac{q^\ell - 1}{q - 1}$$

$$(c) \quad \chi(\mathcal{A}, t) = (t - 1)(t - q) \cdots (t - q^{\ell-1})$$

$$(d) \quad \chi(\mathcal{A}, q^{\ell-1}) = 0.$$

**Proof.** (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) is trivial. (d) $\Rightarrow$ (c) is from the above lemma.  $\square$

Here we assume  $\ell = 3$ , and give some combinatorial characterization for freeness.

**Lemma 9** *Let  $\mathcal{A}$  be an essential free arrangement in  $\mathbb{F}_q^3$  with exponents  $(1, d_2, d_3)$ . If  $d_2 \leq d_3$ , then  $d_2 \leq q$ .*

**Proof.** Note that  $d_2$  is the minimal degree of the logarithmic vector field  $\delta \in D(\mathcal{A})$  which does not a polynomial multiple of the Euler vector field  $\delta_0$  (see Example 3 for the notation). Since  $\delta_1$  is contained in  $D(\mathcal{A})$ , such a minimal degree can not be greater than  $q = \deg \delta_1$ .  $\square$

**Theorem 10** *Let  $\mathcal{A}$  be an arrangement in  $\mathbb{F}_q^3$ .*

- (1) *If  $\chi(\mathcal{A}, q) = 0$ , then  $\mathcal{A}$  is free with exponents  $(1, q, |\mathcal{A}| - q - 1)$ .*
- (2) *If  $|\mathcal{A}| \geq 2q$  and  $\mathcal{A}$  is free then  $\chi(\mathcal{A}, q) = 0$ .*

**Proof.** (1) Suppose that  $\chi(\mathcal{A}, q) = 0$ . We prove the freeness by descending induction on  $|\mathcal{A}|$ . Recall that  $\mathcal{A} = \mathcal{A}_{all}$  is free with exponents  $(1, q, q^2)$  (see Example 3). In general, choose a hyperplane  $H$  which is not a member of  $\mathcal{A}$ . Then by Lemma 5, we have

$$0 = \chi(\mathcal{A}, q) \geq \chi(\mathcal{A} \cup \{H\}, q) \geq 0,$$

hence  $\chi(\mathcal{A} \cup \{H\}, q) = 0$ . From the inductive hypothesis,  $\mathcal{A} \cup \{H\}$  is free with exponents  $(1, q, |\mathcal{A}| - q)$ . Then Theorem 1 enable us to conclude that  $\mathcal{A}$  is free.

(2) Suppose  $\mathcal{A}$  is free. Note that from the assumption on  $|\mathcal{A}|$ ,  $\mathcal{A}$  is an essential arrangement. Hence the characteristic polynomial is of the form  $\chi(\mathcal{A}, t) = (t-1)(t-d_2)(t-d_3)$  with integers  $d_2 \leq d_3$  which satisfy  $d_2 + d_3 = |\mathcal{A}| - 1 \geq 2q - 1$ . From the Lemma 9, we have  $d_2 \leq q \leq d_3$ . However  $d_2 < q < d_3$  contradicts the Lemma 5 (i)  $\chi(\mathcal{A}, q) \geq 0$ , we have either

$$\begin{aligned} (d_2, d_3) &= (q, |\mathcal{A}| - q - 1) && \text{if } |\mathcal{A}| > 2q, \text{ or} \\ (d_2, d_3) &= (q-1, q) && \text{if } |\mathcal{A}| = 2q. \end{aligned}$$

$\square$

By a similar argument to (1), we have the following theorem for higher dimensional cases.

**Theorem 11** *Let  $\mathcal{A}$  be an arrangement in  $\mathbb{F}_q^\ell$ . If  $\mathcal{A}$  satisfies*

$$\chi(\mathcal{A}, q^{\ell-2}) = 0,$$

*then  $\mathcal{A}$  is free with exponents  $(1, q, \dots, q^{\ell-2}, |\mathcal{A}| - 1 - q - \dots - q^{\ell-2})$ .*

**Remark 12** The argument used in the proof of Theorem 10 (1) can be considered as an example of “supersolvable resolution” in [Zi].

In the next result we will treat the cases  $|\mathcal{A}| = 2q - 1$  and  $2q - 2$ .

**Theorem 13** Suppose that  $\chi(\mathcal{A}, q) \neq 0$ .

- (1) When  $|\mathcal{A}| = 2q - 1$ ,  $\mathcal{A}$  is free if and only if  $\chi(\mathcal{A}, t) = (t - 1)(t - q + 1)^2$ .
- (2) When  $|\mathcal{A}| = 2q - 2$ ,  $\mathcal{A}$  is free if and only if  $\chi(\mathcal{A}, t) = (t - 1)(t - q + 1)(t - q + 2)$ .

**Proof.** (1) The similar arguments above from the fact  $\chi(\mathcal{A}, q) \geq 0$  shows that the freeness of  $\mathcal{A}$  implies  $\chi(\mathcal{A}, t) = (t - 1)(t - q + 1)^2$ . Conversely, suppose  $\chi(\mathcal{A}, t) = (t - 1)(t - q + 1)^2$ , then

$$|M(\mathcal{A})| = \chi(\mathcal{A}, q) = q - 1.$$

This means that there exists a line  $L \subset \mathbb{F}_q^3$  such that

$$M(\mathcal{A}) = L \setminus \{0\}.$$

Choose a hyperplane  $H$  containing  $L$ , then  $\mathcal{A} \cup \{H\}$  is free with exponents  $(1, q - 1, q)$ . Again by using Theorem 1, we conclude that  $\mathcal{A}$  is free with exponents  $(1, q - 1, q - 1)$ . (2) can be proved similarly.  $\square$

We can summarize the results as follows.

**Corollary 14** (1) When  $|\mathcal{A}| \geq 2q$ ,  $\mathcal{A}$  is free if and only if  $\chi(\mathcal{A}, q) = 0$ , or equivalently,  $M(\mathcal{A}) = \emptyset$ .

(2) When  $|\mathcal{A}| = 2q - 1$ ,  $\mathcal{A}$  is free if and only if either  $\chi(\mathcal{A}, q) = 0$  or  $\chi(\mathcal{A}, t) = (t - 1)(t - q + 1)^2$ .

(3) When  $|\mathcal{A}| = 2q - 2$ ,  $\mathcal{A}$  is free if and only if either  $\chi(\mathcal{A}, q) = 0$  or  $\chi(\mathcal{A}, q) = (t - 1)(t - q + 1)(t - q + 2)$ .

**Example 15** ([Zi])

Let us fix a line  $L \subset \mathbb{F}_3^3$ , and define an arrangement  $\mathcal{A}_1$  by

$$\mathcal{A}_1 = \{H \in \mathcal{A}_{all}(\mathbb{F}_3^3) \mid H \not\ni L\}.$$

We can easily seen that  $|\mathcal{A}_1| = 9 > 2 \times 3$ . Since  $M(\mathcal{A}_1) \neq \emptyset$ ,  $\mathcal{A}_1$  is not free. However, in [Zi], Ziegler proved the following. Let  $\mathcal{A}_2$  be an arrangement in  $\mathbb{K}^3$  satisfying  $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$ . If the characteristic of the field  $\mathbb{K}$  is not 3, then  $\mathcal{A}_2$  is free with exponents  $(1, 4, 4)$ .

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